sg-Interior and sg-Closure in Topological spaces

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Abstract: In this paper, we introduce sg-interior, sg-closure and some of its basic properties.

Keywords: sg-open; sg-closed; sg-int(A); sg-cl(A); sg-Hausdorff space.

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1. Introduction and Preliminaries

Levine [6] introduced generalized closed sets in topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Many researchers like Arya et al [2], Balachandran et al [3], Bhattarcharya et al [4], Arockiarani et al [1], Gnanambal [5] Malghan [7], Nagaveni [8] and Palaniappan et al [9] have worked on generalized closed sets. In this paper, the notion of sg-interior is defined and some of its basic properties are investigated. Also we introduce the idea of sg-closure in topological spaces using the notions of sg-closed sets and obtain some related results.

Throughout the paper, X and Y denote the topological spaces (X,τ) and (Y,σ) respectively and on which no separation axioms are assumed unless otherwise explicitly stated.

Definition 1.1 A subset A of a space X is called

1) A **preopen** set if A $\subseteq$ int(cl(A)) and a **preclosed** if cl(int(A)) $\subseteq$ A

2) A **regular open set** if A = int(cl(A)) and **regular closed set** if A = cl(int(A))

3) A **semi open set** if A $\subseteq$ cl(int(A)) and **semi closed set** if int(cl(A)) $\subseteq$ A

The intersection of all preclosed subsets of X containing A is called pre-closure of A and is denoted by pcl(A)

Definition1.2: A subset A of a space X is called

1) **g-closed set[6]** if if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X

2) **semi generalized closed set** [4] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi open in X.

3) **generalized preclosed set** [7] if clint(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

The complements of the above mentioned closed sets are their respective open sets.

Defintion 1.3: Let X be a topological space and let x $\in$ X. A subset N of X is said to be sg-neighbourhood of x if there exists a sg-open set G such that x $\in$ G $\subset$ N.

2. sg–closure and interior in Topological space.

Definition 2.1: Let A be a subset of X. A point $x \in A$ is said to be sg-interior point of A is A is a sg-neighbourhood of x. The set of all sg-interior points of A is called the sg-interior of A and is denoted by sg-int(A).

Theorem 2.1: If A be a subset of X. Then sg-int(A) = $\cup$ { G : G is a sg-open, G $\subset$ A}.

Proof: Let A be a subset of X.

$x \in$ sg-int(A) $\Leftrightarrow$ x is a sg-interior point of A.

$\Leftrightarrow$ A is a sg-nbhd of point x.

$\Leftrightarrow$ there exists sg-open set G such that $x \in G \subset A$.

$\Leftrightarrow x \in \cup$ {G:G is a sg-open, G $\subset$ A}

Hence sg-int(A) = $\cup$ {G : G is a sg-open, G $\subset$ A}.

Theorem 2.2: Let A and B be subsets of X. Then

(i) sg-int(X) = X and sg-int(φ) = φ

(ii) sg-int(A) $\subset$ A.

(iii) If B is any sg-open set contained in A, then B $\subset$ sg - int(A).

(iv) If A $\subset$ B, then sg-int(A) $\subset$ sg-int(B).

(v) sg-int(sg-int(A)) = sg-int(A).

Proof: (i) Since X and φ are sg open sets,

by Theorem

sg-int(X) = $\cup$ { G : G is a sg-open, G $\subset$ X}

= X $\cup$ {all sg open sets} = X.

(ie) int(X) = X. Since φ is the only sg- open set contained in φ, sg-int(φ) = φ

(ii) Let $x \in$ sg-int(A) $\Rightarrow$ x is a interior point of A.

$\Rightarrow$ A is a nbhd of x.

$\Rightarrow x \in A$.

Thus, $x \in sg-\text{int}(A) \Rightarrow x \in A$.

Hence sg-int(A) $\subset$ A.

(iii) Let B be any sg-open sets such that B $\subset$ A. Let $x \in B$. Since B is a sg-open set contained in A. x is a sg-interior point of A.

(ie) $x \in$ sg-int(A).

Hence B $\subset$ sg-int(A).

(iv) Let A and B be subsets of X such that A $\subset$ B. Let $x \in$ sg-int(A). Then x is a sg-interior point of A and so A is a sg-nbhd of x. Since B $\supset$ A, B is also sg-nbhd of x. $\Rightarrow x \in$ sg-int(B). Thus we have shown that $x \in$ sg-int(A) $\Rightarrow x \in$ sg-int(B).

Theorem 2.3: If a subset A of space X is sg-open, then sg-int(A) =A.

Proof: Let A be sg-open subset of X. We know that sg-int(A) $\subset$ A. Also, A is sg-open set contained in A. From Theorem (iii) A $\subset$ sg-int(A). Hence sg-int(A) = A.
The converse of the above theorem need not be true, as seen from the following example.

Example 2.1: Let X = {a,b,c} with topology

τ ={X, φ, {b},{c},{a,b},{b,c}}. Then sg-O(X) = {X, φ,{a},{b},{c},{a,b},{b,c}}. sg-int({a,c}) ={a} $\cup$ {c} $\cup$ {φ} = {a,c}. But {a,c} is not sg-open set in X.

Theorem 2.4: If A and B are subsets of X, then sg-int(A) $\cup$ sg-int(B) $\subset$ sg-int(A $\cup$ B).

Proof. We know that A $\subset$ A $\cup$ B and B $\subset$ A $\cup$ B.
We have Theorem 2.2
(iv) sg-int(A) $\subset$ sg-int(A $\cup$ B),
sg-int(B) $\subset$ sg-int(A $\cup$ B).
This implies that
sg-int(A) $\cup$ sg-int(B) $\subset$ sg-int(A $\cup$ B).

Theorem 2.5: If A and B are subsets of X, then sg-int(A $\cap$ B) = sg-int(A) $\cap$ sg-int(B).

Proof: We know that A $\cap$ B $\subset$ A and A $\cap$ B $\subset$ B. We have sg-int(A $\cap$ B) $\subset$ sg-int(A) and sg-int(A $\cap$ B) $\subset$ sg-int(B).
This implies that sg-int(A $\cap$ B) $\subset$ sg-int(A) $\cap$ sg-int(B) -----(1)

Again let x $\in$ sg-int(A) $\cap$ sg-int(B). Then x $\in$ sg-int(A) and x $\in$ sg-int(B). Hence x is a sg-int point of each of sets A and B. It follows that A and B is sg-nbhds of x, so that their intersection A $\cap$ B is also a sg-nbhds of x. Hence x $\in$ sg-int(A $\cap$ B). Thus x $\in$ sg-int(A) $\cap$ sg-int(A) implies that x $\in$ sg-int(A $\cap$ B).

Therefore sg-int(A) $\cap$ sg-int(B) $\subset$ sg-int(A $\cap$ B) ------(2)

From (1) and (2),

We get sg-int(A $\cap$ B)=sg-int(A) $\cap$ sg-int(B).

Theorem 2.6: If A is a subset of X, then int(A) $\subset$ sg-int(A).

Proof: Let A be a subset of X.

Let x $\in$ int(A) $\Rightarrow$ x $\in$ $\cup$ {G : G is open, G $\subset$ A}.

$\Rightarrow$ there exists an open set G such that x $\in$ G $\subset$ A.

$\Rightarrow$ there exist a sg-open set G such that x $\in$ G $\subset$ A, as every open set is a sg-open set in X .

$\Rightarrow$ x $\in$ $\cup$ {G : G is sg- open, G $\subset$ A}.

$\Rightarrow$ x $\in$ sg-int(A).

Thus x $\in$ int(A) $\Rightarrow$ x $\in$ sg-int(A). Hence int(A) $\subset$ sg-int(A).

Remark.2.1: Containment relation in the above theorem may be proper as seen from the following example.
Example 2.2: Let X ={a,b,c} with topology τ ={X, φ, {b},{c},{b,c}}. Then sg-O(X)={X, φ ,{b},{c},{a,b},{a,c},{b,c}}.
Let A = {a,b}. Now sg-int(A) = {a,b} and int(A) = {b}. It follows that int(A) $\subset$ sg-int(A) and int(A) $\neq$ sg-int(A).

Theorem 2.7: If A is a subset of X, then g-int(A) $\subset$ sg-int(A), where g-int(A) is given by g-int(A) = $\cup$ {G : G is g-open, G $\subset$ A}.
Proof: Let A be a subset of X.

Let $x \in$ int(A) $\Rightarrow$ $x \in$ $\cup$ {G : G is g-open, G $\subset$ A}.

$\Rightarrow$ there exists a g-open set G such that $x \in$ G $\subset$ A

$\Rightarrow$ there exists a sg-open set G such that $x \in$ G $\subset$ A, as every

g- open set is a sg-open set in X

$\Rightarrow x \in \cup\{G : G$ is sg-open, $G \subset A\}$.

$x \in$ sg-int(A).

Hence g-int(A) $\subset$ sg-int(A).

Remark 2.2: Containment relation in the above theorem may be proper as seen from the following example.

Example 2.3: Let X ={a,b,c} with topology τ ={X, φ, {b},{c},{a,c}}. Then sg-o(X) = { X, φ, {a},{c},{a,b},{a,c},{b,c}}. & g – open (X) = { X, φ,{a},{c},{a,c}}. Let A = {b,c}, sg-int(A) = {b,c} & g-int(A) = {c}. It follows g-int(A) $\subset$ sg-int(A) and g-int(A) $\neq$ sg-int(A) .

Definition 2.2: Let A be a subset of a space X. We define the sg-closure of A to be the intersection of all sg-closed sets containing A. In symbols, sg-cl(A) = $\cap$ {F : A $\subset$ F $\in$ sgc(X)}.

Theorem 2.8: If A and B are subsets of a space X. Then

(i) sg-cl(X) = X and sg-cl(φ) = φ

(ii) A $\subset$ sg-cl(A).

(iii) If B is any sg-closed set containing A, then sg-cl(A) $\subset$ B.

(iv) If A $\subset$ B then sg-cl(A) $\subset$ sg-cl(B).

Proof: (i) By the definition of sg-closure, X is the only sg-closed set containing X. Therefore sg-cl(X) = Intersection of all the sg-closed sets containing X = $\cap$ {X} = X. That is sg-cl(X) = X. By the definition of sg-closure, sg-cl(φ) = Intersection of all the sg-clsed sets containing φ = $\cap$ {φ} = φ. That is sg-cl(φ) = φ.

(ii) By the definition of sg-closure of A, it is obvious that A $\subset$ sg-cl(A).

(iii) Let B be any sg-closed set containing A. Since sg-cl(A) is the intersection of all sg-closed sets containing A, sg-cl(A) is contained in every sg-closed set containing A. Hence in particular sg-cl(A) $\subset$ B.

(iv) Let A and B be subsets of X such that A $\subset$ B. By the definition sg-cl(B) = $\cap$ { F: B $\subset$ F $\in$ sg-c(X)}. If B $\subset$ F $\in$ sg-c(X), then sg-cl(B) $\subset$ F. Since A $\subset$ B, A $\subset$ B $\subset$ F $\in$ sg-c(X), we have sg-cl(A) $\subset$ F. There fore sg-cl(A) $\subset$ $\cap$ {F : B $\subset$ F $\in$ sg-c(X)} = sg-cl(B).

(i.e) sg-cl(A) $\subset$ sg-cl(A).

Theorem 2.9: If A $\subset$ X is sg-closed, then sg-cl(A) = A.

Proof: Let A be sg-closed subset of X. We know that A $\subset$ sg-cl(A). Also A $\subset$ A and A is sg-closed. By theorem (iii) sg-cl(A) $\subset$ A. Hence sg-cl(A) = A.

Remarks 2.3: The converse of the above theorem need not be true as seen from the following example.

Example 2.4: Let X ={a,b,c} with topology τ ={X, φ, {b},{c},{a,b},{b,c}}. Then sg-C(X)={X, φ ,{a},{c},{a,b},{b,c},{a,c}}. sg-cl({b}) ={b}. But {b} is not sg-closed set in X.

Theorem 2.10: If A and B are subsets of a space X, then sg-cl(A $\cap$ B) $\subset$ sg-cl(A) $\cap$ sg-cl(B).

Proof: Let A and B be subsets of X. Clearly A $\cap$ B $\subset$ A and A $\cap$ B $\subset$ B.
By theorem sg-cl(A $\cap$ B) $\subset$ sg-cl(A) and sg-cl(A $\cap$ B) $\subset$ sg-cl(B).
Hence sg-cl(A $\cap$ B) $\subset$ sg-cl(A) $\cap$ sg-cl(B).

Theorem 2.11: If A and B are subsets of a space X then sg-cl(A $\cup$ B)= sg-cl(A) $\cup$ sg-cl(B).

Proof: Let A and B be subsets of X. Clearly A $\subset$ A $\cup$ B and B $\subset$ A $\cup$ B. We have sg-cl(A) $\cup$ sg-cl(B) $\subset$ sg-cl(A $\cup$ B)

----(1)

Now to prove
sg-cl(A $\cup$ B) $\subset$ sg-cl(A) $\cup$ sg- cl(B).

Let x $\in$ sg-cl(A $\cup$ B) and suppose x $\neq$ sg-cl(A) $\cup$ sg- cl(B). Then there exists sg-closed sets A_1 and B_1 with A $\subset$ A_1, B $\subset$ B_1 and x $\neq$ $A_1 \cup B_{1.}$ We have A $\cup$ B $\subset$ $A_1 \cup B_1$ and $A_1 \cup B_1$ is sg-closed set by theorem such that x $\neq$ $A_1 \cup B_{1.}$ Thus x $\neq$ sg-cl(A $\cup$ B) which is a contradiction to x $\in$ sg-cl(A $\cup$ B). Hence sg-cl(A $\cup$ B) $\subset$ sg-cl(A) $\cup$ sg-cl(B)

----(2)

From (1) and (2), we have

sg-cl(A $\cup$ B)= sg-cl(A) $\cup$ sg-cl(B).

Theorem 2.12: For an x $\in$ X, x $\in$ sg-cl(A) if and only if V $\cap$ A $\neq$ φ for every sg-closed sets V containing x.

Proof: Let x $\in$ X and x $\in$ sg-cl(A). To prove V $\cap$ A $\neq$ φ for every sg-open set V containing x. Prove the result by contradiction. Suppose there exists a sg-open set V containing x such that V $\cap$ A = φ. Then A $\subset$ X-V and X-V is sg-closed. We have sg-cl(A) $\subset$ X - V. This shows that x $\neq$ sg-cl(A), which is a contradiction.
Hence V $\cap$ A $\neq$ φ for every sg-open set V containing x.
Conversly, let V $\cap$ A $\neq$ φ for every sg-open set V containing x. To prove x $\in$ sg-cl(A). We prove the result by contradiction. Suppose x $\neq$ sg-cl(A). Then x $\in$ X – F and S – F is sg-open. Also (X–F) $\cap$ A = φ, which is a contradiction.
Hence x $\in$ sg-cl(A).

Theorem 2.13: If A is a subset of a space X, then sg-cl(A) $\subset$ cl(A).

Proof: Let A be a subset of a space S. By the definition of closure, cl(A) = $\cap$ {F: A $\subset$ F $\in$ C(X)}.

If $A \subset F \in C(X)\}$, Then $A \subset F \in$ sg-C(X), because every closed set is sg-closed. That is sg-cl(A) $\subset$ F. There fore sg-cl(A) $\subset \cap \{F \subset X : F \in C(X)\}$ = cl(A).
Hence sg-cl(A) $\subset$ cl(A).

Remark 2.4: Containment relation in the above theorem may be proper as seen from the following example.

Example 2.5: Let X ={a,b,c} with topology τ ={X, φ, {b},{c},{a,c}}. Then sg-cl(X) = {X, φ, {a},{b},{c},{a,b},{b,c}} and g – cl (X) = { X, φ, {b},{a,b},{b,c}}. Let A = {b,c}, sg-cl(A) = {b,c} and g-cl(A) = {b}. It follows g-cl(A) $\subset$ sg-cl(A) and g-cl(A) $\neq$ sg-cl(A) .

Theorem 2.14 : If A is a subset of X, then sg-cl(A) $\subset$ g-cl(A), where g-cl(A) is given by g-cl(A) = $\cap \{F \subset X : A \subset F$ and f is a g-closed set in X}.
Proof: Let A be a subset of X. By definition of g-cl(A) = $\cap \{F \subset X : A \subset F$ and f is a g-closed set in X}. If $A \subset F$ and F is g-closed subset of x, then $A \subset F \in$ sg-cl(X), because every g closed is sg-closed subset in X. That is sg-cl(A) $\subset$ F.
Therefore sg-cl(A) $\subset \cap \{F \subset X : A \subset F$ and f is a g-closed set in X} = g-cl(A).
Hence sg-cl(A) $\subset$ g-cl(A).

Corrolory2.1: Let A be any subset of X. Then

(i) sg-int(A))^c = sg-cl(A^c)

(ii) sg-int(A) = (sg-cl(A^c))

(iii) sg-cl(A) = (sg-cl(A^c))

Proof: Let x $\in$ sg-int(A))^c. Then x $\notin$ sg-int(A). That is every sg-open set U containing x is such that U $\not\subset$ A. That is every sg-open set U containing x is such that $U \cap A^c \neq \varphi$. By theorem x $\in$ sg-int(A))^c and there fore sg-int(A))$^c \subset$ sg-cl(A^c). Conversely, let x $\in$ sg-cl(A^c).
Then by theorem, every sg-open set U containing x is such that U $\cap A^c \neq \varphi$. That is every sg-open set U containing x is such that U $\not\subset$ A. This implies by definition of sg-interior of A, x $\notin$ sg-int(A). That is x $\in$ sg-int(A))^c and sg-cl(A^c) $\subset$ (sg-int(A))^c. Thus sg-int(A))^c = sg-cl(A^c)

(ii) Follows by taking complements in (i).

(ii) Follows by replacing A by A^c in (i).

3. Preservation Theorems Concerning G-Hausdorff and sg-Hausdorff Spaces

In this section we investigate preservation theorems concerning sg-Hausdorff spaces.

Defintion 3.1: A topological space x is said to be g-Hausdorff if whenever x and y are distinct points of X there are disjoint g-open sets U and V with x $\in$ U and y $\in$ V.

It is obvious that every Hausdorff space is g-Hausdorff space. The following example shows that the converse is not true.

Example 3.1: Let X = {a,b,c} and τ ={X, φ, {a}}. It is clear that X is not Hausdorff Space. Since {a}, {b} and {c} are all g-open, it follows that H is sg-Hausdorff Space.

Theorem3.1: Let X be a topological space and Y be Hausdorff. If f: X $\rightarrow$ Y is injective and g-continous, then x is g-Hausdorff.
Proof: Let x and y be any two distinct points of X. Then f(x) and f(y) are distinct points of Y, because f is injective. Since Y is Hausdorff, there are disjoint open sets U and V in Y containing f(x) and f(y) respectively. Since f is g-continous and U $\cap$ V = φ, we have f^{-1}(U) and f^{-1}(V) are disjoint g-open sets in X such that x $\in$ f^{-1}(U) and y $\in$ f^{-1}(V). Hence X is g-Hausdorff space.

Defintion3.2: A topological space X is said to be sg-Hausdorff Space if whenever x and y are distinct points of X there are disjoint sg-open sets U and V with x $\in$ U and y $\in$ V.

It is obvious that every g-Hausdorff space is a sg-Hausdorff space. The following example shows that the converse is not true.

Example 3.1: Let X = {a,b,c} and τ ={X, φ, {a}}. Since {a}, {b} and {c} are all sg-open, it implies that X is sg-Hausdorff space. Since {a}, {b} and {c} are not g-open in X , it follows that 'a' and 'c' can not be separated by any two disjoint g-open sets in X. Hence X is not g-Hausdorff Space.

Theorem3.2: Let X be a topological space Y be Hausdorff space. If f: X $\rightarrow$ Y is injective and sg-continuous, then X is sg-Hausdorff Space.

Proof: Let x and y be any two distinct points of X. Then f(x) and f(y) are distinct points of Y, because f is injective. Since Y is Hausdorff, there are disjoint open sets U and V in Y containing f(x) and f(y) respectively. Since f is sg-continous and U $\cap$ V= φ, we have f^{-1}(U) and f^{-1}(V) are disjoint sg-open sets in X such that x $\in$ f^{-1}(U) and y $\in$ f^{-1}(V). Hence X is sg-Hausdorff space.

Theorem3.3: Let X be a topological space Y be sg-Hausdorff Space. If f: X $\rightarrow$ Y is injective and sg-irresolute, then X is sg-Hausdorff space.
Proof: Let x and y be any two distinct points of X. Then f(x) and f(y) are distinct points of Y, because f is injective. Since Y is sg-Hausdorff, there are disjoint sg- open sets U and V in Y containing f(x) and f(y) respectively. Since f is sg-irresolute and U $\cap$ V = φ, we have f^{-1}(U) and f^{-1}(V) are disjoint sg-open sets in X such that x $\in$ f^{-1}(U) and y $\in$ f^{-1}(V). Hence X is sg-Hausdorff space.

4. Conclusion

From the definitions of g-Hausdorff space and sg-Hausdorff space, we have result.

X is a Hausdorff Space $\Rightarrow$ X is a g- Hausdorff Space $\Rightarrow$ X is a sg- Hausdorff Space.

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